Regular and Singular Orthants of Tridiagonal Matrices

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ABSTRACT

With respect to a tridiagonal matrix with variable diagonal vector g, an orthant is said to be regular (singular) if the matrix is nonsingular (singular) for all g in it. We give necessary and sufficient conditions for an orthant to be regular or singular. Our idea is based on observations of a simple two-by-two matrix, and all the results obtained are original and self-contained.

1. INTRODUCTION

This paper is concerned with tridiagonal matrices of the form

$$A(g)=\begin{bmatrix} g(1) & 1 & & & \\ 1 & g(2) & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & g(n-1) & 1 \\ & & & 1 & g(n) \end{bmatrix}.$$

We shall treat such a matrix as a function of its diagonal vector $g = (g(1),\dots,g(n))$. Our results were originally motivated by the observation that open quadrants exist in the plane such that the simple tridiagonal matrix

$$\begin{bmatrix} x & 1 \\ 1 & y \end{bmatrix}$$

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is nonsingular for every vector (x, y) in these quadrants. Indeed, since this simple matrix is singular if and only if $xy = 1$, they are exactly the second and the fourth quadrants. This observation will be extended to the more general matrix $A(g)$ in the following sections. Here we introduce a few terminologies to facilitate later discussions.

We let $\operatorname{sign} x$ be defined by

$$\operatorname{sign} x = \begin{cases} +1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x < 0. \end{cases}$$

For g and p in R^n, g is equivalent to p $(g \simeq p)$ if

$$\operatorname{sign} g(k) = \operatorname{sign} p(k), \qquad 1 \leqslant k \leqslant n.$$

This relation $\simeq$ is an equivalence relation, and defines a partition into 3^n equivalence classes. An equivalence class will be denoted by $\Omega(g)$ and called the orthant containing g. Let S be a set of vectors g in R^n; we say that S is *regular* [with respect to a tridiagonal matrix $B(g)$ with diagonal vector g] if $B(g)$ is nonsingular for all g in S. We shall be concerned with the conditions which characterize the regular orthants.

Let $A_k(g)$ be the determinant of the kth (leading) principal matrix of $A(g)$. For the sake of simplicity, we shall write A_k instead of $A_k(g)$ if the dependence of $A_k(g)$ on g is not essential. By means of the Lagrange method for evaluating determinants, we see that A_k is given by the three-term recurrence relation

$$A_{k+1} = g(k+1)A_k - A_{k-1}, \qquad k = 1, 2, \ldots, n-1 \qquad (*)$$

where $A_0 = 1$ and $A_1 = g(1)$.

We remark that we cannot have $A_{i-1} = A_i = 0$ for any i, $1 \leqslant i \leqslant n$. For otherwise by $(*)$, $A_k = 0$ for $1 \leqslant k \leqslant n$. But then $A_1 = g(1) = 0$ and $A_2 = g(2)g(1) - 1 = 0$, which is impossible. This fact will be used in the proof of Lemma 5.

2. PRELIMINARIES

In view of the recurrence relation $(*)$, it is plausible that appropriate sign patterns for the diagonal entries $g(k)$ (regardless of its magnitude) may be so chosen that $g(k)A_{k-1}$ is of the same weak sign as A_k. If care is also taken in case some of the diagonal entries are actually zero, we may further have $g(k)A_{k-1}A_k > 0$. The details of these ideas are contained in the following four lemmas.

LEMMA 1. *Let i and j be two integers satisfying $1 \leqslant i < j \leqslant n$. Suppose $g(k)g(k+1) < 0$ for $i \leqslant k \leqslant j-1$. If $g(k)A_{k-1}A_k > 0$ for $k = i$, then $g(k)A_{k-1}A_k > 0$ for $i+1 \leqslant k \leqslant j$.*

Proof. Assume by induction that our assertion holds for $k = j-1$, then by $(*)$,

$$\begin{aligned} g(j)A_{j-1}A_j &= g(j)A_{j-1}\left[g(j)A_{j-1} - A_{j-2}\right] \\ &= g^2(j)A_{j-1}^2 - \frac{[g(j-1)g(j)]\left[g(j-1)A_{j-2}A_{j-1}\right]}{g^2(j-1)} \\ &> 0, \end{aligned}$$

as desired. ∎

LEMMA 2. *Let i and j be two integers satisfying $1 \leqslant i < j \leqslant n$. Suppose $g(k) = 0$ for $i+1 \leqslant k \leqslant j-1$. If $g(k)A_{k-1}A_k > 0$ for $k = i$ and if $(-1)^{i+j}g(i)g(j) \geqslant 0$, then $g(k)A_{k-1}A_k \geqslant 0$ for $k = j$, where strict inequality holds if and only if $g(j) \neq 0$.*

Proof. In view of $(*)$, $A_k = -A_{k-2}$ for $i+1 \leqslant k \leqslant j-1$. Since $A_{i-1} \neq 0$ and $A_i \neq 0$, we see by induction that $A_k \neq 0$ for $i+1 \leqslant k \leqslant j-1$. Hence $A_{i+1}A_{i+2} \cdots A_{j-1} = [-A_{i-1}] \cdots [-A_{j-3}]$ or $A_{j-2}A_{j-1} = (-1)^{i+j+1}A_{i-1}A_i$. As a consequence,

$$g(j)A_{j-1}A_j = g^2(j)A_{j-1}^2 - (-1)^{j+1-i}\frac{g(i)g(j)[g(i)A_{i-1}A_i]}{g^2(i)} \geqslant 0,$$

where strict inequality holds iff $g(j) \neq 0$. ∎

LEMMA 3. *Let i and j be odd integers satisfying $1 \leqslant i < j \leqslant n+1$. Suppose $g(k) = 0$ for $k = i+2, i+4, \ldots, j-2$. If $A_{i-1} \neq 0$ and $A_i = 0$, then $A_k = 0$ for $k = i+2, i+4, \ldots, j-2$ and $A_k \neq 0$ for $k = i+1, i+3, \ldots, j-1$. Moreover, if $g(j) \neq 0$ then $g(j)A_{j-1}A_j > 0$.*

Proof. Note that since

$$A_{i+1}=g(i+1)A_i-A_{i-1}=-A_{i-1}\neq 0,$$

$$A_{i+2}=g(i+2)A_{i+1}-A_i=0,\ldots,$$

we have $A_k=0$ for $k=i+2,i+4,\ldots,j-2$ and $A_k=(-1)^{(k-i+1)/2}A_{i-1}\neq 0$ for $k=i+1,i+3,\ldots,j-1$. Note further that $A_j=g(j)A_{j-1}$; thus if $g(j)\neq 0$ then $g(j)A_{j-1}A_j=g^2(j)A_{j-1}^2>0$. ■

LEMMA 4. *Let i and m be two positive integers satisfying $1\leqslant i$ and $i+2m\leqslant n$. Suppose $g(k)=0$ for $k=i+2,i+4,\ldots,i+2m$. If $A_i\neq 0$, then $A_k\neq 0$ for $k=i+2,i+4,\ldots,i+2m$.*

Proof. The proof follows from $A_{i+2}=-A_i\neq 0,\ldots,\quad A_{i+2m}=-A_{i+2m-2}\neq 0$. ■

LEMMA 5. *Let i and j be two integers satisfying $1\leqslant i<j\leqslant n$. Suppose $g(k)=0$ for $i+1\leqslant k\leqslant j-1$. If $g(i)\neq 0$, and $A_i(g)=0$, then there exist vectors $u=(u(1),\ldots,u(n))$ and $v=(v(1),v(2),\ldots,v(n))$ such that*

$$\operatorname{sign} g(k)=\operatorname{sign} u(k)=\operatorname{sign} v(k),\qquad k\neq j,$$

$$u(j)v(j)<0,$$

and $A_j(u)=A_j(v)=0$.

Proof. We leave $u(j)$ and $v(j)$ to be determined and let

$$u(k)=\begin{cases} g(i)/2, & k=i,\\ g(k), & k\neq i,j,\end{cases}$$

and

$$v(k)=\begin{cases} 2g(i), & k=i,\\ g(k), & k\neq i,j.\end{cases}$$

Since $A_{i-1}(u) \neq 0$ (see the remark at the end of Section 1) and

$$A_i(u) = \frac{g(i)A_{i-1}(u)}{2} - A_{i-2}(u) = \frac{g(i)A_{i-1}(u)}{2} - g(i)A_{i-1}(u)$$

$$= \frac{-g(i)A_{i-1}(u)}{2},$$

$$A_{i+1}(u) = -A_{i-1}(u), \ldots, \qquad A_{j-1}(u) = -A_{j-3}(u),$$

we have

$$\frac{A_{k-1}(u)}{A_k(u)} = \frac{A_{k-3}(u)}{A_{k-2}(u)}, \qquad i+2 \leqslant k \leqslant j-1,$$

$$\frac{A_i(u)}{A_{i+1}(u)} = \frac{g(i)}{2}, \qquad \frac{A_{i-1}(u)}{A_i(u)} = \frac{-2}{g(i)},$$

from which we calculate that

$$\frac{A_{j-2}(u)}{A_{j-1}(u)} = \begin{cases} \dfrac{-2}{g(i)}, & j-i \text{ odd}, \\ \dfrac{g(i)}{2}, & j-i \text{ even}. \end{cases}$$

Similarly,

$$\frac{A_{j-2}(v)}{A_{j-1}(v)} = \begin{cases} \dfrac{1}{g(i)}, & j-i \text{ odd}, \\ -g(i), & j-i \text{ even}. \end{cases}$$

If we now set $u(j)$ equal to $A_{j-2}(u)/A_{j-1}(u)$ and $v(j)$ to $A_{j-2}(v)/A_{j-1}(v)$, then clearly $u(j)v(j) < 0$, $A_j(u) = u(j)A_{j-1}(u) - A_{j-2}(u) = 0$, and $A_j(v) = v(j)A_{j-1}(v) - A_{j-2}(v) = 0$ as desired. ■

We remark that in the above lemma, j can be equal to $i+1$. In this case, the condition that $g(k) = 0$ for $i+1 \leqslant k \leqslant j+1$ is taken to be vacuous.

LEMMA 6. *Let i and j be two integers satisfying $1 \leqslant i < j \leqslant n$. Suppose $g(k) = 0$ for $i+1 \leqslant k \leqslant j-1$. If $g(k)A_{k-1}(g)A_k(g) > 0$ for $k = i$ and if $(-1)^{i+j}g(i)g(j) < 0$, there then exists a vector $u = (u(1), u(2), \ldots, u(n))$ such that $u \simeq g$ and $A_i(u) = 0$.*

Proof. We leave $u(j)$ to be determined and let $u(k) = g(k)$ for $k \neq j$. Since $u(k)A_{k-1}(u)A_k(u) > 0$ for $k = i$ and $A_k(u) = -A_{k-2}(u)$ for $i+1 \leqslant k \leqslant j-1$, we see that

$$\frac{A_{j-2}(u)}{A_{j-1}(u)} = \begin{cases} \dfrac{A_{i-1}(u)}{A_i(u)}, & j-i \text{ odd}, \\[2ex] \dfrac{A_i(u)}{A_{i-1}(u)}, & j-i \text{ even}. \end{cases}$$

If we let $u(j) = A_{j-2}(u)/A_{j-1}(u)$, then $A_j(u) = 0$; moreover, it is easily seen that $(-1)^{i+j}u(i)u(j) < 0$. This implies $\operatorname{sign} u(j) = \operatorname{sign} g(j)$ as required. ∎

We remark that in the above lemma, j can be equal to $i+1$. In this case, this lemma asserts that if $g(i)g(i+1) > 0$ and $g(k)A_{k-1}(g)A_k(g) > 0$ for $k = i$, then there exists a vector $u \simeq g$ and $A_i(u) = 0$.

3. MAIN RESULTS

For any $g = (g(1), \ldots, g(n))$ in R^n, let $\hat{g}(k)$, $0 \leqslant k \leqslant n+1$, be defined by

$$\hat{g}(k) = \begin{cases} 1, & k = 0, n+1, \\ g(k) & \text{otherwise}. \end{cases}$$

Now let

$$\alpha(g) = \inf\{k = 1 + 2m \mid m = 0, 1, 2, \ldots,\ 0 \leqslant k \leqslant n+1,\ \hat{g}(k) \neq 0\}$$

and let

$$\beta(g) = \sup\{k = n - 2m \mid m = 0, 1, 2, \ldots,\ 0 \leqslant k \leqslant n+1,\ \hat{g}(k) \neq 0\}.$$

For examples, if $g=(0,0)$, then $\hat{g}(0)=1$, $\hat{g}(1)=0$, $\hat{g}(2)=0$, $\hat{g}(3)=1$, and $\alpha=3$, $\beta=0$; if $g=(0,0,0)$, then $\alpha=\infty$ and $\beta=-\infty$; if $g=(1,1)$, then $\alpha=1$ and $\beta=2$. If α and β are integers such that $\alpha>\beta$, then we define $S[\alpha,\beta]$ to be empty. If $1\leqslant\alpha\leqslant\beta\leqslant n$, we define

$$S[\alpha,\beta]=\{k|\alpha\leqslant k\leqslant\beta \text{ and } g(k)\neq 0\}.$$

In general, $S[\alpha,\beta]$ can be written as the union of disjoint integral intervals $S_1, S_2, \dots, S_r$ where

$$\alpha=\min S_1(\leqslant \max S_1),$$

$$1+\max S_i<\min S_{i+1}, \qquad 1\leqslant i\leqslant r-1,$$

$$(\min S_r\leqslant)\max S_r=\beta.$$

For convenience, we shall denote $\min S_i$ by $m(i)$ and $\max S_i$ by $M(i)$.

THEOREM 1. *Suppose $\alpha=\alpha(g)$ and $\beta=\beta(g)$ are finite. Suppose $S[\alpha,\beta]=S_1\cup S_2\cup\cdots\cup S_r$, where $\alpha=m(1)$, $M(i)+1<m(i+1)$ for $1\leqslant i\leqslant r-1$ and $M(r)=\beta$. Suppose*

(i) *$g(k)g(k+1)<0$ for $m(i)\leqslant k<M(i)$, $1\leqslant i\leqslant r$, and*
(ii) *$(-1)^{m(i+1)+M(i)}g(M(i))g(m(i+1))>0$ for $1\leqslant i<r$.*

Then $A_n(g)\neq 0$.

Proof. Assume first that $\alpha\geqslant\beta$. Suppose $\alpha-\beta=2m+1$ for some non-negative integer m. According to Lemma 3, $A_{\alpha-1}A_\alpha\neq 0$. Also,

$$g(\beta+2m+2)=g(\beta+2m+4)=\cdots=g(n)=0$$

and

$$A_{\alpha+1}=A_{\beta+2m+2}=g(\beta+2m+2)A_\alpha-A_{\alpha-1}=-f(\alpha)h(\alpha)A_{\alpha-1}\neq 0.$$

Thus by Lemma 4, $A_n\neq 0$. Suppose $\alpha-\beta$ is even. Then $\alpha=\beta$ by their definitions and $g(\alpha+2)=\cdots=g(n)=0$. By Lemma 3, $A_\alpha\neq 0$, so that by Lemma 4, $A_n\neq 0$. Now let $\alpha<\beta$. By Lemma 3, $g(\alpha)A_{\alpha-1}A_\alpha>0$. By Lemma 1 and hypothesis (i), $g(k)A_{k-1}A_k>0$ for $k=M(1)$. By Lemma 2 and hypothesis (ii), $g(k)A_{k-1}A_k>0$ for $k=m(2)$. Inductively, we have $g(\beta)A_{\beta-1}A_\beta>0$. Finally, by Lemma 4, $A_n\neq 0$. This completes the proof. ∎

THEOREM 2. *$\Omega(g)$ is regular with respect to $A(g)$ if and only if g satisfies the hypotheses of Theorem* 1.

Proof. We need to show that if g does not satisfy the hypotheses of Theorem 1, then there exists a vector $u \simeq g$ and $A_n(u) = 0$. If α is not finite, then n must be odd and $g(1) = g(3) = \cdots = g(n) = 0$. Since $A_n(g) = 0$ by Lemma 3, we may take $u = g$. Similarly, we may take $u = g$ if β is not finite. Thus we may assume α and β are both finite numbers. Now let λ be

$$\inf\{k | g(k)g(k+1) > 0,\ m(i) \leqslant k < M(i),\ 1 \leqslant i \leqslant r\},$$

and τ be the infimum of the set of integers i for which condition (ii) is violated. Note that λ cannot be equal to $M(\tau)$ by their definitions. Also, by our assumptions, $\min\{\lambda, M(\tau)\}$ exists. If $M(\tau)$ is equal to $\min\{\lambda, M(\tau)\}$, then (as seen in the proof of Theorem 1) we have $g(k)A_{k-1}(g)A_k(g) > 0$ for $k = M(\tau)$. By Lemma 6, we see that there is some $v \simeq g$ such that $A_k(v) = 0$ for $k = m(\tau + 1)$. Now we can apply Lemma 5 repeatedly to conclude that $A_\beta(w) = 0$ for some $w \simeq g$. Thus we can see from $A_\beta(w) = 0$, $A_{\beta+2}(w) = -A_\beta(w) = 0, \ldots,$ $A_n(w) = -A_{n-2}(w) = 0$ that $u = w$ is the desired vector.

If $\lambda = \min\{\lambda, M(\tau)\}$, then (as seen in the proof of Theorem 1) we have $g(k)A_{k-1}(g)A_k(g) > 0$ for $k = \lambda$. Now we are in a situation similar to the previous case. As a consequence, we may repeat the same arguments to conclude that $A_n(u) = 0$ for some $u \simeq g$. This completes the proof. ∎

So far we have dealt with the regular orthants. It is natural to define a *singular* orthant $\Omega(g)$ to be one such that $A(g)$ is singular for all g in $\Omega(g)$, and ask for a characterization of it. This is a relatively easy problem. If n is even, then we assert that no singular orthant can exist. Suppose to the contrary that $\Omega(p)$ is singular. Let p_j be a sequence of vectors in $\Omega(p)$ converging to the zero vector. Then $A_n(p_j) = 0$ for each j. Also, by continuity of $A_n(\cdot)$, $A_n(0) = 0$. But by Lemma 3, $A_n(0) \neq 0$ when n is even. This contradiction concludes our proof. Next, consider the case when n is odd. Note that if $g(1) = g(3) = \cdots = g(n) = 0$, then $A_n(g) = 0$ by Lemma 3.

Conversely, we assert that if $g(i) \neq 0$ for some odd integer i, then $\Omega(g)$ cannot be singular. To see this, suppose to the contrary that $\Omega(g)$ is singular. Let p_j be the sequence of vectors defined by

$$p_j(k) = \begin{cases} g(i), & k = i, \\ g(k)/j, & k \neq i. \end{cases}$$

Then $p_j \simeq g$, so that $A_n(p_j) = 0$. Since p_j converges to $p' =$

$(0,\ldots,0, g(i), 0,\ldots,0)$, we have $A_n(p')=0$. But by Lemma 3, $A_{i-1}(p')A_i(p')\neq 0$, which implies, in view of Lemma 4, that $A_n(p')\neq 0$. This contradiction concludes our proof. We summarize these statements as follows.

THEOREM 3. *$\Omega(g)$ is singular if and only if n is odd and $g(1)=g(3)=\cdots=g(n)=0$.*

4. CONSEQUENCES AND CONCLUSIONS

There are some interesting consequences of Theorem 3. Suppose $g(k)\neq 0$ for $1\leqslant k\leqslant n$; then $\alpha(g)=1$, $\beta(g)=n$, and $S[\alpha(g),\beta(g)]$ is equal to $\{1,2,\ldots,n\}$. Thus by Theorem 2, if $g(k)\neq 0$ for $1\leqslant k\leqslant n$, then $\Omega(g)$ is regular iff $g(k)g(k+1)<0$ for $1\leqslant k\leqslant n$. This result is a generalization of the observation stated in the first section. As another example, when $n=3$, there are exactly 3 singular orthants, 16 regular orthants, and 8 orthants which are neither regular nor singular.

The above results can be extended to the more general tridiagonal matrix of the following form:

$$B(g)=\begin{bmatrix} g(1) & h(1) & & & \\ f(2) & g(2) & h(2) & & \\ & \ddots & \ddots & \ddots & \\ & & & f(n-1) & g(n) \end{bmatrix},$$

where $f(k)h(k)\neq 0$ for $1\leqslant k\leqslant n$. The technique for doing so is probably well known and can be inferred from the following sequence of operations:

$$\det B(g)=h(1)\det\begin{bmatrix} g(1)/h(1) & 1 & & \\ f(2) & g(2) & h(2) & \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

$$=f(1)h(1)\det\begin{bmatrix} g(1)/h(1)f(1) & 1 & & \\ 1 & g(2) & h(2) & \\ & \ddots & \ddots & \ddots \end{bmatrix}=\cdots.$$

Using the notation in Section 3, the above technique will enable us to derive the following

THEOREM 4. *$\Omega(g)$ is regular with respect to $B(g)$ if and only if $\alpha(g)$ and $\beta(g)$ are finite and*

(i) $\operatorname{sign}[g(k)g(k+1)] = \operatorname{sign}[-f(k)h(k)]$ *for* $m(i) \leqslant k < M(i)$, $1 \leqslant i \leqslant r$, *and*
(ii) $\operatorname{sign}[g(M(i))g(m(i+1))] = \operatorname{sign}[(-1)^{m(i+1)+M(i)}\prod_{k=M(i)}^{m(i+1)-1} f(k)h(k)]$, $1 \leqslant i < r$.

We remark that Theorem 4 can be restated in the following form:

THEOREM 5. *$\Omega(g)$ is regular with respect to $B(g)$ if and only if $\alpha(g)$, $\beta(g)$ are finite and*

$$(-1)^{i+j} g(i)g(j) \prod_{k=i}^{j-1} f(k)h(k) \geqslant 0.$$

for any i, j satisfying $\alpha(g) \leqslant i < j \leqslant \beta(g)$.

The equivalence of these two formulations is easy to see; the latter can also be proved directly by means of Lemmas 1 through 6.

Finally, the assumptions that $f(k)h(k) \neq 0$ for $1 \leqslant k \leqslant n-1$ can be relaxed further. If $f(k)h(k) = 0$ for some j, and $f(k)h(k) \neq 0$ for $k \neq j$, then $B_n(g)$ is equal to the product of the determinants of two submatrices of $B(g)$. As a consequence, the regularity of $\Omega(g)$ is equal to the simultaneous regularity of two orthants of lower dimension, and the singularity is equivalent to either one of these two orthants being singular. The general case where $f(k)h(k) = 0$ for more than one k can be dealt with similarly.

5. ADDITIONAL REMARKS

We are indebted to the referee for informing us that there is an extensive literature (see [1, 2, 3, 4] and the references therein) concerning with sign solvability of linear systems which is related to the subject matter discussed in our paper. More specifically, it is known [2] that the study of sign solvability can be decomposed into the study of L-matrices and S-matrices, where A is a L-matrix (or sign-nonsingular matrix) if every matrix with the same sign pattern as A is nonsingular. In this terminology, our Theorem 5 is a characterization of tridiagonal L-matrices. There are characterizations of

L-matrices [1; 2, Remark 1.1]. However, they are algorithmic or graph-theoretic in nature and are different from ours. In view of the facts and methods already developed, we expect to see generalizations and related results in the near future.

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